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TEMPERATURE FIELD OF A MOVING POINT SOURCE WITH CHANGE OF STATE

N. D. MALMUTH
Science Center, Rockwell International, P.O. Box 1085, Thousand Oaks, CA 91360, U.S.A.

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Abstract—A theoretical simulation of automatic welding as a moving point source of heat with conduction as the dominant mode of transfer is refined to incorporate change of state across the melt boundary. Matched asymptotic expansions are used to solve the singular perturbation problem associated with arbitrary latent heat, and small values of source dimensionless velocity—power product. The problem is mathematically analogous to scattering of long waves off a sphere. It is shown that previous empirical estimates which subtract the latent heat power from the corresponding input quantity seriously over-predict the effect of change of state on weld depth.

NOMENCLATURE*

C.P., common part;
D, thermal diffusivity;
g_n, equation (3a);
h_n, equation (11);
i, j, k, unit vectors along Cartesian axes (Fig. 1);
k, thermal conductivity;
L, Helmholtz operator, equation (1a), latent heat;
(r, θ, ϕ), spherical polar coordinates (Fig. 1);
V, travel speed;
$(\bar{x}, \bar{y}, \bar{z})$, dimensional Cartesian frame traveling with source (Fig. 1);
(x, y, z), dimensionless Cartesian coordinates.

Greek symbols

α, speed-power parameter;
γ, latent heat parameter;
λ, diffusion length, $2D/V$;
ρ, dimensionless polar radius from source point, mass density;
ρ_i, dimensionless interfacial position;
τ, reduced temperature;
τ_i, interfacial temperature;
Δ, Laplacian operator;
[ ], jump in a quantity across interface.

Subscript

i, interface.

Superscripts

+, solid side of interface;
−, liquid side of interface.

INTRODUCTION

A NUMBER of problems in machining and joining processes involve heat addition through the action of a cutting tool or a heat torch. This leads to important transformations in the metallurgical and mechanical properties of the workpiece. To predict these changes, knowledge of the associated thermal response is required.

Most welding processes exemplify these ideas. The description of the associated temperature fields has been a subject of intensive research, see for example, Rosenthal [1] in regard to workpiece thermal response, and [2–7] in connection with the determination of the heat flux distribution associated with freely burning arcs of welding type. These distributions are subject to a large variety of complex hydromagnetic processes involving transient three dimensional turbulent flows with coexistence of the processed material in all of the four states of matter. Nevertheless, progress can be made when the spatial scales of these phenomena are small compared to the fusion zone dimensions. Such circumstances frequently occur in laser, plasma, and arc welding, and lead to the assumption that the welding torch can be characterized by an intense moving source of heat, [1, 8], with the extent of the fusion zone primarily determined by conduction processes, and the convection and radiation playing a secondary role in controlling the strength of the source. Although this model leads to useful conclusions regarding the parametric dependence of thermal cycles, peak temperatures, and weld geometries, the role of certain nonlinear processes such as change of state, must be determined to establish the range of applicability of the simpler linear model. For welding, the phase transformations correspond to freezing and vaporization of a molten "pool" of metal below the heat source. The energy stored in the latent heat of fusion for steels can be as much as 30 per cent of that required to bring the material to its melting point, and the corresponding quantity for vaporization can be ten times higher. These factors underscore the need for quantitative information. Unfortunately, such information must come from the solution of a nonlinear free-boundary value problem. Problems of this type have been studied in one dimension in an extensive number of physical

*Numbers refer to equation in text where symbol is defined.

situations. An account of these is given in [8]. Treatment of three dimensional systems involving simultaneous energy addition and phase change seems to be limited to numerical methods such as those described in [9]. To obtain information on parametric dependencies, some sort of analytical solution is required. Although the mathematical problem shows no encouraging prospects for exact solution, the application of perturbation methods suggests the possibility of practically useful approximate solutions. Such representations were obtained in [10, 11] corresponding to a single phase boundary associated with a fusion. That analysis can be readily generalized to handle additional free boundaries such as those for vaporization. In [10] and [11], the approximate developments were based on formal asymptotic expansions of limit process type [12], involving two lumped parameters arising in the non-dimensional formulation of the problem. Denoting $\bar{\rho}$ as the density, k as the conductivity,* L = latent heat of fusion, T_m = melting temperature,† V = lineal speed of welding torch, (see Fig. 1), c = specific heat, D = thermal diffusivity = $k/\bar{\rho}c$, Q = net heat flux, these parameters are a latent heat parameter $\gamma \equiv L/cT_m$, and a power-speed parameter $\alpha \equiv QV/4\pi k T_m D$. The first two terms of the appropriate expansion for the temperatures and interface contour were obtained in [10, 11] for the limit $\gamma \to 0$, with α and the dimensionless polar radius ρ (see Fig. 1), fixed. The latter is expressed in units of the characteristic diffusion length $\lambda = 2D/V$. In this expansion, the dominant term corresponds to vanishing γ and the second term is the approximate perturbation associated with change of state.

It is of interest to investigate the temperature field when the parameter γ is no longer small. For this purpose, asymptotic expansions will be treated in this paper for the limit $\alpha \to 0$, γ fixed. In contrast to the regular expansion about $\gamma \to 0$ previously discussed, this limit leads to a singular perturbation problem for reasons that will be made clear. Uniformly valid representations will be obtained for the dominant term of the asymptotics for the phase change boundary and the temperature field outside of it. To make this paper self contained, the formulation of the dimensionless temperature problem will be reproduced here from [10, 11].

FORMULATION OF EXACT BOUNDARY VALUE PROBLEM

Referring to Fig. 1, the point source is depicted as lying at the origin of the Cartesian set $(\bar{x}, \bar{y}, \bar{z})$ or the spherical polar set $(\bar{r}, \theta, \phi)$. The source simulates the welding torch which is stationary in the frame, so that the material moves relative to the torch with the velocity, $-V\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the $\bar{z}$ direction as shown in Fig. 1. The workpiece is assumed

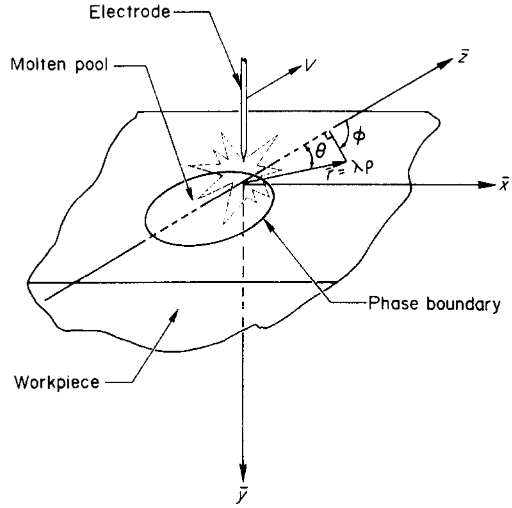

FIG. 1. Geometry of problem.

to be the infinite half space $y > 0$. Furthermore, only one phase boundary will be considered for the present, and without loss of generality will be designated as that corresponding to fusion. The case of two boundaries, i.e. vaporization and fusion, will be deferred to a subsequent analysis. Denoting T as the temperature elevation above ambient conditions (occurring at infinity), the appropriate boundary value problem can be formulated in terms of the following normalized variables

$$x \equiv \bar{x}/\lambda,\quad y \equiv \bar{y}/\lambda,\quad z \equiv \bar{z}/\lambda,\quad \rho \equiv \bar{r}/\lambda,\quad \tau \equiv Te^{z}/T_m.$$

Conservation of energy then leads to the following statement of the exact problem, in which the plane $y = 0$ is considered adiabatic, as specified in (1g):

$$(\Delta - 1)\tau \equiv L[\tau] = 0 \tag{1a}$$

$$\lim_{\rho \to 0} \rho^2 \tau_\rho = -\alpha \tag{1b}$$

$$[\nabla\tau \,.\, \mathbf{n}]_i = 2\gamma\tau_i \mathbf{k}\,.\,\mathbf{n},\quad [\;]_i \equiv (\;)_{\rho_{i+}} - (\;)_{\rho_{i-}} \tag{1c}$$

$$[\tau]_i = 0 \tag{1d}$$

$$\tau_i = e^{\rho_i \cos\theta} \tag{1e}$$

$$\tau = o(e^{\rho\cos\theta}) \quad \text{as} \quad \rho \to \infty \tag{1f}$$

$$\left.\frac{\partial\tau}{\partial\phi}\right|_{\phi=0} = 0 \tag{1g}$$

where

$$\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

and ρ_i represents the phase change interfacial boundary. In what follows, the region $\rho \leqslant \rho_i$ will be referred to as the "molten pool" in conformance to welding applications.

SMALL POWER-ARBITRARY LATENT HEAT APPROXIMATION

In this section, the method of matched asymptotic expansions [12–14], is applied to obtain the limiting representation of the temperature field and phase boundary at small power-velocity levels ($\alpha \to 0$). These conditions imply nearly circular isotherms, with the

*As in [10, 11], a constant value is assigned for this parameter.

†Without excessive loss of generality, the thermal properties of the liquid and solid states are assumed here to coincide.

temperatures being approximately those produced by a stationary welding torch. In the present units, the limiting solution corresponds to the melting isotherm, $\rho_i(\theta)$ shrinking to the source point, i.e. $\rho_i = O(\alpha)$ as $\alpha \to 0$. To keep the interface boundary ρ_i in view as $\alpha \to 0$, a strained variable $\tilde{\rho}$ and an "inner" limit are defined in the following manner:

$$\tilde{\rho} \equiv \frac{\rho}{\alpha} \quad \text{fixed as} \quad \alpha \to 0. \tag{2}$$

Thus, the interface position expressed in the new units, $\tilde{\rho}_i \equiv \rho_i/\alpha = O(1)$ as $\alpha \to 0$, and the "fine structure" of the molten pool $\rho \leqslant \rho_i$ can be analyzed in the $\alpha \to 0$ limit. Problems of this type arise in other areas of continuum mechanics, the most famous application being in fluid dynamics, where a coordinate stretching is required to see the details of the "boundary layer" or region of important viscous effects (near the surface of a body) in a slightly viscous flow. Another situation which leads to a boundary value problem analogous to (1) but with real frequency, is the scattering of a plane wave from a sphere whose radius is small compared to the wave length of the incoming wave. In this, as well as the present case, there are two disparate length scales. For the scattering problem, these are the wave length of the incoming wave and the radius of the sphere, and in the present case, are the diffusion length $2D/V$ and the melting radius at $V = 0$, namely $Q/2\pi k T_m$. This situation usually implies a singular perturbation problem, in which for an assumed parametric limit, approximate representations of the solution are not uniformly valid in the space of the independent variables. Prescriptions for uniformizing the solution have been devised for special boundary value problems for ordinary differential equations, but no comprehensive theory exists for partial differential equations.

For the case at hand, it is asserted that in the limit (2), the asymptotic representations of the reduced temperature, τ, and the phase boundary ρ_i are

$$\tau(\rho, \theta;\, \alpha, \gamma)_{\text{inner}} \doteq g_0(\tilde{\rho}) + \sum_1^{\infty} \alpha^n g_n(\tilde{\rho}, \theta;\, \gamma) \tag{3a}$$

$$\rho_i(\theta;\, \alpha, \gamma) \doteq \alpha\left[1 + \sum_1^{\infty} \alpha^n R_n(\theta;\, \gamma)\right] \tag{3b}$$

where γ is fixed and $g_n, R_n = O(1)$ as $\alpha \to 0$.

The zeroth order inner boundary value problem resulting from substitution of (3) into (1) and retention of the dominant terms is

$$(\tilde{\rho}^2 g_0')' = 0 \qquad \left(' \equiv \frac{\mathrm{d}}{\mathrm{d}\tilde{\rho}}\right) \tag{4a}$$

$$\lim_{\tilde{\rho} \to 0} \tilde{\rho}^2 g_0' = -1 \tag{4b}$$

$$[g_0']_i = 0 \tag{4c}$$

$$[g_0]_i = 0 \tag{4c}$$

$$g_{0_{i,}} \equiv g_0(1) = 1. \tag{4d}$$

Approximation of the dominant term of the $\gamma \to 0$ expansion in [10, 11] in the inner limit (2) gives

$$g_0 = \rho^{-1} \tag{5}$$

which satisfies all the conditions of the problem (4).

Retention of the next order terms in the previously-mentioned substitution process gives the following first order problem:

$$\tilde{\Delta} g_1 = 0 \tag{6a}$$

$$\tilde{\rho}^2 \tilde{\Delta} \equiv \frac{\partial}{\partial \tilde{\rho}}\left(\tilde{\rho}^2 \frac{\partial}{\partial \tilde{\rho}}\right) + \operatorname{cosec}\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)$$

$$\left[\frac{\partial g_1}{\partial \tilde{\rho}}\bigg|_i\right] = 2\gamma \cos\theta \tag{6b}$$

$$[g_{1_i}] = 0 \tag{6c}$$

$$R_1 = g_{1i} - \cos\theta \tag{6d}$$

In (6) as well as (4), the conditions at the point at infinity have been left unspecified. The reason for this will become clear presently. Denoting quantities with $(-)$ and $(+)$ superscripts as those corresponding to $\rho < \rho_i$ and $\rho > \rho_i$, respectively, it is reasonable to assume in view of (6b) and the theory of harmonic surface distributions [15–17] that the solution of (6) can be expressed in terms of the first two Laplace spherical harmonics. The steps justifying this assertion follow arguments given in [11]. Thus,

$$g_1^{\pm} = \frac{A_{10}^{\pm}}{\tilde{\rho}} + B_{10}^{\pm} + \left(\frac{A_{11}^{\pm}}{\tilde{\rho}} + B_{11}^{+}\tilde{\rho}\right)\cos\theta. \tag{7}$$

Substitution of (7) into (6b), (6c), yields

$$g_1^- = B_{10}^+ + \left(B_{11}^+ - \frac{2\gamma}{3}\right)\tilde{\rho}\cos\theta \tag{8a}$$

$$g_1^+ = B_{10}^+ + \left(B_{11}^+\tilde{\rho} - \frac{2\gamma}{3\tilde{\rho}^2}\right)\cos\theta. \tag{8b}$$

It thus appears that g_1 is undetermined; there seeming to be no other relations to determine the B_{1n}^+. At this point, it is recalled that the conditions at infinity have not been specified for the zeroth and first order boundary value problems. The reason for this omission is associated with a non-uniformity at the point at infinity for g_1. For this purpose, consider the limiting solution for τ as $\gamma \to 0$ given by τ_0 in [10, 11], where

$$\left.\begin{aligned} \tau_0 &= o(\tilde{\Delta}\tau_0), & \rho &= O(\alpha) \\ \text{and} \quad \tau_0 &= \Delta\tau_0, & \rho &= O(1) \end{aligned}\right\} \quad \text{as} \quad \alpha \to 0 \qquad \begin{aligned} &(9a) \\ &(9b) \end{aligned}$$

i.e. τ_0 is approximately a zeroth order Laplace harmonic for small enough radius, since $e^{-\rho}$ is an entire function. Moreover, the essential singularity in this function at the point at infinity implies a divergent Taylor's expansion in that neighborhood, nullifying the validity of the Laplace harmonic approximation, and requiring a spherical Bessel harmonic representation in its stead. Since τ_0 is a limiting solution, (9) should apply for $\alpha \to 0$, with $\gamma \neq 0$, i.e. with the zero subscripts removed.

To formalize the preceding intuitive arguments, consider an "outer" limit defined as:

$$\rho, \gamma \text{ fixed as } \alpha \to 0. \tag{10}$$

With this choice of coordinates, the pool, $\rho < \rho_i$ will shrink to the origin, and the appropriate asymptotic expansion is asserted to be:

$$\tau^+(\rho,\theta;\,\alpha,\gamma)_{(\text{outer})} \doteq h_0(\rho)+\sum_1^\infty \alpha^n h_n(\rho,\theta;\,\gamma),\qquad h_n = O(1). \tag{11}$$

On substitution of (11) into (1), retaining only the dominant term

$$(\Delta-1)h_0 = 0 = \left\{\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^2\frac{\mathrm{d}}{\mathrm{d}\rho}\right]-1\right\}h_0$$

and thus:

$$h_0 = C(\alpha)\frac{\mathrm{e}^{-\rho}}{\rho}. \tag{12}$$

Here, the means of determining the constants in (12) and (8) becomes evident; i.e. since (12) and (8) are representations of presumably the same function in different regions, they could perhaps be mutually valid in some overlap domain, denoted as

$$\rho = O(\eta(\alpha)) \quad \text{as} \quad \alpha\to 0 \tag{13}$$

where η is an order class such that

$$O(\alpha) < O(\eta(\alpha)) < O(1). \tag{14}$$

In this overlap or "intermediate" region, the representations (11) and (3) should "match" to give identical representations to all orders of α. It is this matching condition that determines the unknown constants in the inner and outer solutions. Before formalizing the matching condition, it should be noted that often the inner and outer expansions do not match to each other but rather to an intermediate expansion. This more involved case appears not to occur in this problem.

To construct the representations of the inner and outer solutions, define an intermediate limit, consistent with (13) and (14) as

$$\rho_\eta \equiv \frac{\rho}{\eta(\alpha)} \equiv \frac{\alpha\tilde{\rho}}{\eta(\alpha)} \quad \text{fixed as} \quad \alpha\to 0. \tag{15}$$

Let

$$\lim_\eta \equiv \lim_{\alpha\to 0\ \rho_\eta\ \text{fixed}} \tag{16}$$

Formalizing the previous discussion, the matching condition to order α^n can be written as:

$$\lim_\eta \frac{\tau^+_{\text{outer}}(\rho_\eta,\theta;\,\alpha,\gamma)-\tau^+_{\text{inner}}(\rho_\eta,\theta;\,\alpha,\gamma)}{\alpha^n} = 0 \tag{17}$$

providing τ_{outer} and τ_{inner} match directly in the manner discussed above.

Now, to apply (17), the representation in intermediate variables of the inner and outer solutions is required. From (5), (8), and (12), these are

$$\tau^+_{\text{inner}} \doteq \frac{\alpha}{\eta\rho_\eta}+\alpha\left[B^+_{10}+\frac{\eta\rho_\eta}{\alpha}B^+_{11}\cos\theta\right]+\ldots \tag{18a}$$

$$\tau^+_{\text{outer}} \doteq C(\alpha)\left[\frac{1}{\eta\rho_\eta}-1+\ldots\right]+\ldots. \tag{18b}$$

To the order of approximation to be considered here, only zeroth order matching is required. Thus, on applying (17)

$$\lim_\eta(\tau^+_{\text{outer}}-\tau^+_{\text{inner}}) = \left\{C(\alpha)\left[\frac{1}{\eta\rho_\eta}-1+\ldots\right]\right\} - \left\{\frac{\alpha}{\eta\rho_\eta}+\alpha\left[B^+_{10}+\frac{\eta\rho_\eta}{\alpha}B^+_{11}\cos\theta\right]\right\} = 0. \tag{19}$$

It is clear from (19) that

$$C = \alpha \tag{20a}$$

$$B^+_{10} = -1 \tag{20b}$$

$$B^+_{11} = 0. \tag{20c}$$

For later purposes, denote the common part (C.P.) as the terms that cancel in the matching. This is found from (19) to the order of approximation to be

$$\text{C.P.} \doteq \tilde{\rho}^{-1}-\alpha+\ldots. \tag{21}$$

Substitution of (20b) and (20c) into (8) gives for the inner expansion

$$\tau = \tau^- = \tilde{\rho}^{-1}-\alpha\left(1+\frac{2\gamma}{3}\tilde{\rho}\cos\theta\right)+\ldots,\quad \tilde{\rho}<\tilde{\rho}_i \tag{22a}$$

$$= \tau^+ = \tilde{\rho}^{-1}-\alpha\left(1+\frac{2\gamma}{3}\tilde{\rho}^{-2}\cos\theta\right)+\ldots,\quad \tilde{\rho}>\tilde{\rho}_i. \tag{22b}$$

The phase boundary is obtained from application of (22) to (6d). Noting that

$$-g^-_1 = 1+\frac{2\gamma}{3}\tilde{\rho}\cos\theta$$

$$-g^+_1 = 1+\frac{2\gamma}{3}\tilde{\rho}^{-2}\cos\theta$$

it follows that

$$-R_1 = 1+\left(\frac{2\gamma}{3}+1\right)\cos\theta \tag{23a}$$

$$\tilde{\rho}_i = 1-\alpha\left[1+\left(1+\frac{2\gamma}{3}\right)\cos\theta\right]+\ldots. \tag{23b}$$

The outer expansion is merely the dominant term of the $\gamma\to 0$, α fixed expansion, i.e. substitution of (20a) into (12) gives

$$\tau^+_{\text{outer}} \doteq h_0 = \frac{\alpha\,\mathrm{e}^{-\rho}}{\rho}. \tag{24}$$

A uniformly valid representation of τ^+ can be obtained as a composite expansion by adding the inner and outer expansions and subtracting the common part. Thus, from (22b), (24), and (21)

$$\tau^+_{\text{composite}} = \tau^+_{\text{inner}}+\tau^+_{\text{outer}}-\text{C.P.} = \alpha\left[\frac{\mathrm{e}^{-\rho}}{\rho}-\frac{2\gamma}{3\tilde{\rho}^2}\cos\theta\right]+\ldots \tag{25}$$

which is consistent with $\alpha\to 0$ limit of the appropriate small γ expansion given in [10, 11].

DISCUSSION

Of primary interest in welding applications is the "penetration", or vertical extent of the fusion zone. To a good approximation, this may be estimated to be the maximum half width of the melting isotherm. The effect of latent heat on this dimension at small power levels can be obtained from (23b) for $\alpha \to 0$. The angle and radius at which the pool width is a maximum are respectively

$$\theta_{\max} = \frac{\pi}{2} + \alpha\theta_1(\gamma) + \dots \qquad (26a)$$

$$\tilde{\rho}_{\max} = 1 + \alpha R_1\left(\frac{\pi}{2} + O(\alpha)\right). \qquad (26b)$$

Denoting this width by $\tilde{y}$, it follows that

$$\begin{aligned}\tilde{y} &= \tilde{\rho}_{\max}\sin\theta_{\max} \\ &= \left[1+\alpha R_1\left(\frac{\pi}{2};\gamma\right)\right]\left[\sin\frac{\pi}{2}+\alpha\theta_1\cos\frac{\pi}{2}+O(\alpha^2)\right] \qquad (27) \\ &\doteq 1+\alpha R_1\left(\frac{\pi}{2};\gamma\right)+\dots.\end{aligned}$$

But from (23a)

$$R_1\left(\frac{\pi}{2};\gamma\right) = R_1\left(\frac{\pi}{2};0\right) = -1. \qquad (28)$$

Substitution of (28) and (26) yields finally

$$\tilde{y} = 1 - \alpha + \dots. \qquad (29)$$

Thus, the latent heat has an effect on the penetration that is $O(\alpha^2)$ as $\alpha \to 0$. This behavior is in agreement with calculations based on a numerical integral equation solution developed in [18].

A quantitative validation of the solutions embodied in (22) and (23) is given in Table 1 and Fig. 2, where the results for the interfacial boundary position using these relations are compared with those given in [18] for $\alpha = 0{\cdot}01$, $\gamma = 10$, and $\alpha = 0{\cdot}1$, $0{\cdot}2$ at $\gamma = 1$. For $\alpha = 0{\cdot}01$, the agreement is excellent over most of the range.

Table 1. Comparison of melting isotherm predictions from solution of equation (23b) and [18] for $\alpha = 0{\cdot}01$, $\gamma = 10$

θ (rad.)	$\tilde{\rho}_i$ (Section 6)	$\tilde{\rho}_i$ [18]
0	0·9133	0·9208
0·1953	0·9148	0·9217
0·4478	0·9209	0·9264
0·7639	0·9314	0·9345
0·9550	0·9457	0·9458
1·2078	0·9628	0·9602
1·4598	0·9815	0·9769
1·7109	1·0007	0·9952
1·9611	1·0192	1·0142
2·2104	1·0358	1·0325
2·4588	1·0495	1·0486
2·7064	1·0595	1·0611
2·9521	1·0653	1·0687
π	1·0667	1·0703

In Fig. 2, good agreement between the asymptotic and numerical solutions is shown, away from the head of the melting isotherm, $\theta = 0$. For $\gamma = 1$, this α value appears to be near the limit of usefulness of the two term expansions, as exemplified by the large discrepancies evident for $\alpha = 0{\cdot}2$. An inspection of (23b) shows

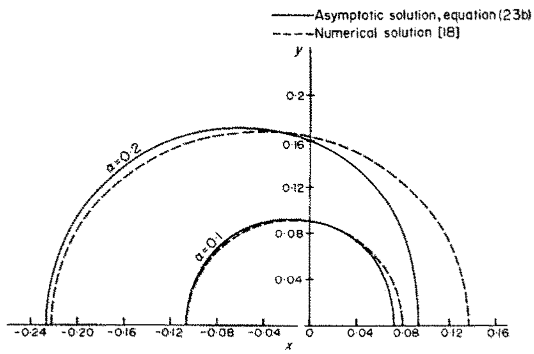

FIG. 2. Comparison of numerical and asymptotic solutions for head and tail boundaries, $\alpha = 0{\cdot}1$.

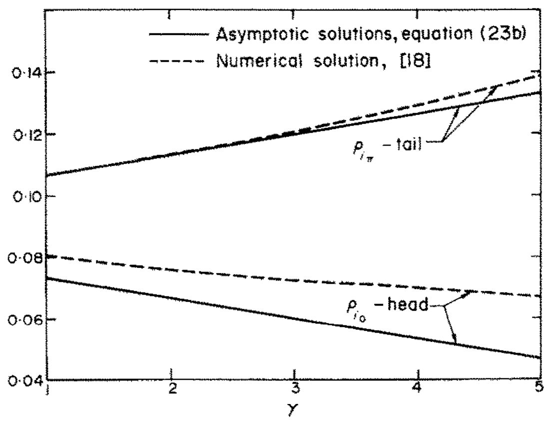

FIG. 3. Comparison of numerical and asymptotic solutions for phase boundaries, $\gamma = 1$.

that the largest perturbations occur in the vicinity of $\theta = 0$. In fact, for $\gamma = 0$, the head radius ρ_{i_0} is $1-\alpha$, as compared with the tail value, ρ_{i_π}, which is $1+O(\alpha^2)$, making the increased sensitivity near the nose of the isotherm plausible. This aspect is corroborated in Fig. 3 which compares the head and tail isotherm radii, ρ_{i_0} and ρ_{i_π} asymptotic results against those of [18] for various γ at $\alpha = 0{\cdot}1$. It is evident that greater discrepancies occur at the head than at the tail. Predictably, these differences also increase with γ.

In particular, it should be noted that the foregoing results are non-uniformly valid for $\gamma \to \infty$, as $\alpha \to 0$. This can easily be seen from (22), (23) and (25). For this limit, other perturbation procedures may be feasible.

The present solution is particularly significant in the light of past efforts by other workers to account for the latent heat effects by empirically subtracting the associated power from the input value. Using the previous notation, the mass flow across a section of the fusion zone of width d is $\rho V d L$, implying the net power input to be $Q-e$. It is seen from (29) that this procedure is too crude an approximation to reality, since it ignores the liberation of energy at the tail of the pool due to freezing. Based on the present analysis, the decrement in penetration will certainly be overpredicted by the empirical method.

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CHAMP DE TEMPERATURE AUTOUR D'UNE SOURCE PONCTUELLE MOBILE AVEC CHANGEMENT D'ETAT

Résumé—On améliore une simulation numérique du soudage automatique pour laquelle la source ponctuelle mobile de chaleur est associée au mode prédominant de transfert par conduction, afin d'introduire le changement d'état à travers la couche de fusion. On utilise des développements asymptotiques appropriés pour résoudre le problème de perturbation singulier, associé à une chaleur latente arbitraire et à de faibles valeurs du produit vitesse-puissance rendu adimensionnel. Le problème est mathématiquement analogue à celui de la dispersion d'ondes longues autour d'une sphère. On montre que les estimations empiriques qui retranchent la puissance d'origine latente de la puissance introduite surrestiment fortement l'effet du changement d'état sur la profondeur de la soudure.

TEMPERATURFELD EINER BEWEGTEN PUNKTQUELLE MIT PHASENÄNDERUNG

Zusammenfassung—Der Vorgang des automatischen Schweißens wird theoretisch simuliert durch eine bewegte Wärmequelle mit Wärmeableitung als dem dominierenden Einfluß und es wird hier der Einfluß der Phasenänderung auf die Schmelzgrenze in die Betrachtung einbezogen. Angepaßte asymptotische Näherungen dienen zur Lösung des singulären Störungsproblems mit beliebiger Latentwärme und kleinen Werten der dimensionslosen Quellgeschwindigkeit. Das Problem ist mathematisch analog der Ausbreitung langer Wellen von einer Kugel. Es wird gezeigt, daß die frühere empirische Abschätzung, wonach die Latentwärme von der entsprechenden zugeführten Wärme abzuziehen ist, zu einer starken Überschätzung des Einflußes der Phasenänderung auf die Schweißtiefe führt.

ТЕМПЕРАТУРНОЕ ПОЛЕ ПЕРЕМЕЩАЮЩЕГОСЯ ТОЧЕЧНОГО ИСТОЧНИКА ПРИ ФАЗОВЫХ ПРЕВРАЩЕНИЯХ

Аннотация — Теоретическое представление автоматической сварки как движущегося точечного источника тепла с доминированием теплопроводности в процессе переноса тепла усовершенствована путем учета изменения агрегатного состояния вдоль границы расплава. Для решения задачи сингулярных возмущений при произвольном значении скрытой теплоты и небольших значениях произведения безразмерной скорости источника на энергию используются двойные асимптотические разложения. Математически задача аналогична случаю рассеяния длинных волн от сферы. Показано, что предыдущие эмпирические оценки, когда энергия скрытой теплоты вычиталась из количества поступившей теплоты, значительно превышают влияние изменения состояния по глубине сварочного шва.